Indian Statistical Institute, Bangalore
B.Math (Hons.) I Year, Second Semester
Mid-Semestral Examination

Time: 3 hours | 29 Feb 2012 | Instructor: C.R.E.Raja

Analysis II | Maximum marks: 40

Answer any five questions, each question is worth 8 marks

1. (i) A sequence (u_n) converges in $\mathbb{R}^k$ if and only if $(u_n(i))$ converges in $\mathbb{R}$ for $1 \leq i \leq k$ where $u_n = (u_n(1), u_n(2), \cdots, u_n(k))$ for $n \geq 1$.

   (ii) Prove that every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

2. (i) If a Cauchy sequence (x_n) in a metric space X has a convergent subsequence, then (x_n) converges.

   (ii) Let (E_n) be a decreasing sequence of closed subsets of a complete metric space X such that $\text{diam}(E_n) > 0$ and $\text{diam}(E_n) \to 0$. Then prove that $\cap E_n$ has exactly one point.

3. (i) Let E be a subset of a metric space X. Then $x \in \overline{E}$ if and only if there is a sequence (x_n) in E such that $x_n \to x$.

   (ii) A metric space X is compact if and only if every collection $\{E_i \mid i \in I\}$ of closed sets that has finite intersection property satisfies $\cap E_i \neq \emptyset$.

4. (i) Let $E \subset \mathbb{R}^k$. If every infinite subset of E has a limit point in E, then prove that E is closed.

   (ii) Prove that every interval in $\mathbb{R}$ is connected.

5. (i) Let $f, g\colon X \to \mathbb{R}$ be continuous functions. Show that $f \pm g$ is continuous and $\{x \in X \mid f(x) \neq g(x)\}$ is an open subset of X.

   (ii) Let $f\colon X \to Y$ be continuous and $\{E_n \mid n \geq 1\}$ be a decreasing sequence of compact subsets of X. Prove that $f(\cap E_n) = \cap f(E_n)$.

6. Let A be a subset of a metric space (X, d). Define $f_A\colon X \to \mathbb{R}$ by
$$f_A(x) = \inf_{a \in A} d(x, a)$$
   for any $x \in X$.

   (i) Prove that f_A is continuous on X.

   (ii) If A is compact, then prove that for each $x \in X$ there is a $a_x \in A$ such that $f_A(x) = d(x, a_x)$.

[P.T.O]

7. (i) If $f\colon X \to Y$ is uniformly continuous and (x_n) is a Cauchy sequence in X, then prove that $(f(x_n))$ is a Cauchy sequence in Y.

   (ii) Prove that continuous function on a compact metric space is uniformly continuous.